# The asymptotic behaviour of recurrence coefficients for orthogonal polynomials with varying exponential weights

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**Abstract**

We consider orthogonal polynomials $\{p_{n,N}(x)\}_{n=0}^{\infty}$ on the real line with respect to a weight $w(x) = e^{-NV(x)}$ and in particular the asymptotic behaviour of the coefficients $a_{n,N}$ and $b_{n,N}$ in the three term recurrence $x\pi_{n,N}(x) = \pi_{n+1,N}(x) + b_{n,N}\pi_{n,N}(x) + a_{n,N}\pi_{n-1,N}(x)$. For one-cut regular $V$ we show, using the Deift-Zhou method of steepest descent for Riemann-Hilbert problems, that the diagonal recurrence coefficients $a_{n,n}$ and $b_{n,n}$ have asymptotic expansions as $n \to \infty$ in powers of $1/n^2$ and powers of $1/n$, respectively.

## 1 Introduction

We consider the asymptotic behavior of the recurrence coefficients $a_{n,N}$ and $b_{n,N}$ in the three-term recurrence relation

$$x\pi_{n,N}(x) = \pi_{n+1,N}(x) + b_{n,N}\pi_{n,N}(x) + a_{n,N}\pi_{n-1,N}(x)$$

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for orthogonal polynomials with respect to varying exponential weights. Here $\pi_{n,N}$ is the $n$-th degree monic orthogonal polynomial with respect to a varying weight

$$w_N(x) = e^{-NV(x)}$$

where $V$ is real analytic on $\mathbb{R}$ with $\lim_{x\to\pm\infty} \frac{V(x)}{\log(1+x^2)} = +\infty$. Moreover, $V$ is assumed to be one-cut regular, which means that the equilibrium measure $d\mu_V = \psi_V(x)dx$ associated with $V$ is supported on one interval $[a, b]$ where it has the form

$$\psi_V(x)\,dx = \sqrt{(b-x)(x-a)}h(x)\chi_{[a,b]}(x)\,dx \tag{1.1}$$

where $h$ is real analytic, strictly positive on $[a, b]$, and in addition the inequality (3.1) is strict for $x \in \mathbb{R} \setminus [a, b]$. See e.g. [1, 2, 5, 11, 17] for the definition of the equilibrium measure and for more information on the one-cut regular case.

Under these assumptions Deift et al. [7] proved that $a_{n,n}$ and $b_{n,n}$ have asymptotic expansions in powers of $1/n$. Their approach is based on the Deift-Zhou method of steepest descent applied to the Riemann-Hilbert problem for orthogonal polynomials of Fokas, Its, and Kitaev [12]. This method was first introduced in [9] and further developed in [6, 7, 8] and many papers since then.

The asymptotic result on the recurrence coefficients was considerably refined by Bleher and Its [2, Theorem 5.2] who showed for polynomial $V$ that there exists $\varepsilon > 0$ and real analytic functions $f_{2k}(s)$, $g_{2k}(s)$, $k = 0, 1, \ldots$, on $[1-\varepsilon, 1+\varepsilon]$ such that the asymptotic expansions

$$a_{n,N} \sim f_0\left(\frac{n}{N}\right) + \sum_{m=1}^{\infty} N^{-2m} f_{2m}\left(\frac{n}{N}\right) \tag{1.2}$$

$$b_{n,N} \sim g_0\left(\frac{n+1/2}{N}\right) + \sum_{m=1}^{\infty} N^{-2m} g_{2m}\left(\frac{n+1/2}{N}\right) \tag{1.3}$$

hold uniformly as $n, N \to \infty$ with $1-\varepsilon \leq n/N \leq 1+\varepsilon$. These $1/N^2$ expansions are used in [2] to prove the $1/N^2$ expansion of the free energy (a.k.a. logarithm of the partition function or Hankel determinant) of the associated random matrix ensemble in the one-cut regular case, see also [11].

The proof of (1.2) and (1.3) in [2] is based on the Deift et al. result referred to above, in combination with so-called string equations. It is of some interest to find a proof that is based on the Riemann-Hilbert steepest

descent analysis only. Here we do this for the diagonal case $n = N$, and we obtain the following.

**Theorem 1.1.** Let $V$ be real analytic and one-cut regular. Then there exist constants $\alpha_{2m}$ and $\beta_m$, $m = 1, 2, \ldots$ (depending on $V$) such that $a_{n,n}$ and $b_{n,n}$ have the following asymptotic expansions as $n \to \infty$:

$$a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{\alpha_{2m}}{n^{2m}}, \qquad b_{n,n} \sim \frac{b+a}{2} + \sum_{m=1}^{\infty} \frac{\beta_m}{n^m}, \tag{1.4}$$

where $a$ and $b$ are the endpoints of the support of $\psi_V$. The first coefficient $\beta_1$ in the expansion for $b_{n,n}$ is given explicitly by

$$\beta_1 = \frac{1}{2\pi(b-a)} \left( \frac{1}{h(b)} - \frac{1}{h(a)} \right) \tag{1.5}$$

where $h$ is the function appearing in the expression (1.1) for the equilibrium measure $\psi_V$ associated with $V$.

In our proof of Theorem 1.1 we follow the main lines of the steepest descent analysis of [7]. We will deduce that the odd powers in the expansion of $a_{n,n}$ vanish from the structure of the local Airy parametrices around the endpoints. The expression (1.5) for $\beta_1$ is new, although it is likely that it can be deduced from the approach of [2] as well. The explicit formula (1.5) shows that $\beta_1 = 0$ if and only if $h(a) = h(b)$. It is very easy to construct examples of one-cut regular $V$ such that $h(a) \neq h(b)$ and so $\beta_1 \neq 0$. We have thus corrected an error in a paper of Albeverio, Pastur, and Shcherbina [1, Theorem 1, formula (1.34)] who claim that $\beta_1 = 0$ always in the one-cut regular case.

**Example 1.2.** We may explicitly check Theorem 1.1 using Jacobi polynomials with varying parameters $\alpha = AN$, $\beta = BN$, $A, B > 0$. These polynomials are orthogonal with weight $(1-x)^{AN}(1+x)^{BN}$ on $[-1, 1]$. The equilibrium measure takes the form (1.1) with

$$a, b = \frac{B^2 - A^2 \pm 4\sqrt{(1+A+B)(1+A)(1+B)}}{(2+A+B)^2} \tag{1.6}$$

and

$$h(x) = \frac{2+A+B}{2\pi(1-x^2)}, \tag{1.7}$$

see [16, 15]. We are in the one-cut regular case, but for weights restricted to $[-1, 1]$. An analysis of the proof of Theorem 1.1, however, will show that the results (1.4)-(1.5) remain valid in this case as well.

From the explicit form of the recurrence coefficients for Jacobi polynomials, see e.g. [4, 15],

$$
\begin{aligned}
a_{n,n} &= \frac{4(1+A+B)(1+A)(1+B)}{((2+A+B)^2-\frac{1}{n^2})(2+A+B)^2} \\
b_{n,n} &= \frac{B^2-A^2}{(2+A+B)(2+A+B+\frac{2}{n})},
\end{aligned}
$$

it is easy to see that (1.4) holds. Using (1.6)-(1.7) we can also ascertain the validity of (1.5).

## 2 The Riemann-Hilbert Problem

The Riemann-Hilbert problem for orthogonal polynomials was introduced by Fokas, Its, and Kitaev [12]. It asks for a $2 \times 2$ matrix valued function $Y(z)$ satisfying

$$
\begin{cases}
Y(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R} \\
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix} \text{ for } x \in \mathbb{R} \\
Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \text{ as } z \to \infty.
\end{cases}
\tag{2.1}
$$

The unique solution of (2.1) is (see e.g. [5])

$$
Y(z) = \begin{pmatrix} \kappa_{n,N}^{-1} p_{n,N}(z) & \frac{1}{2\pi i \kappa_{n,N}} \int\limits_{\mathbb{R}} \frac{p_{n,N}(t)}{t-z} dt \\ -2\pi i \kappa_{n-1,N} p_{n-1,N}(z) & -\kappa_{n-1,N} \int\limits_{\mathbb{R}} \frac{p_{n-1,N}(t)}{t-z} dt \end{pmatrix}
\tag{2.2}
$$

where $p_{n,N}(x) = \kappa_{n,N}\pi_{n,N}(x)$ is the $n$th degree orthonormal polynomial. The recurrence coefficients are expressed as follows in terms of the solution of the Riemann-Hilbert problem (2.1), see [5, 10].

**Proposition 2.1.** Let

$$
Y(z) = \left(I + \frac{1}{z}Y_1 + \frac{1}{z^2}Y_2 + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}
\tag{2.3}
$$

Then

$$
a_{n,N} = (Y_1)_{12}\,(Y_1)_{21}
\tag{2.4}
$$

and

$$b_{n,N} = \frac{(Y_2)_{12}}{(Y_1)_{12}} - (Y_1)_{22} \tag{2.5}$$

For the remainder of this paper we will take $N = n$. We closely follow [5, 7] in applying the Deift-Zhou method of steepest descent for Riemann-Hilbert problems to (2.1).

# 3 The Deift-Zhou method of steepest descent

The goal of the Deift-Zhou method of steepest descent for Riemann-Hilbert problems is to change the original problem into a problem for which the asymptotics for $z \to \infty$ are normalised and for which all matrices, jump matrices and solutions alike, are asymptotically close to the identity matrix for large $n$ which can be solved iteratively. The specific details and steps needed to achieve this goal shall be explained below.

## 3.1 The First Step: Transformation $Y \mapsto T$

The key aspect of the first step of the analysis is the equilibrium measure $\mu_V$ corresponding to $V$. This equilibrium measure $\mu_V$ is the unique probability measure that satisfies for some $l$,

$$2 \int \log |x - y|^{-1} \, d\mu_V(y) + V(x) \geq l, \qquad \text{for all } x \in \mathbb{R}, \tag{3.1}$$

$$2 \int \log |x - y|^{-1} \, d\mu_V(y) + V(x) = l, \qquad \text{for all } x \in \text{supp } \mu_V. \tag{3.2}$$

For the one-cut regular case that we are considering we have that the support is one interval $[a, b]$ and $d\mu_V(x) = \psi_V(x)\, dx$ as in (1.1). In addition the inequality (3.1) is strict for $x \in \mathbb{R} \setminus [a, b]$.

Define

$$g(z) = \int \log(z - s)\, d\mu_V(s) = \int \log(z - s)\psi_V(s)\, ds \tag{3.3}$$

and

$$\phi(z) = \pi \int_b^z ((s - b)(s - a))^{\frac{1}{2}} h(s)\, ds, \quad z \in \mathbb{C} \setminus (-\infty, b] \tag{3.4}$$

$$\tilde{\phi}(z) = \pi \int_a^z ((s - b)(s - a)))^{\frac{1}{2}} h(s)\, ds, \quad z \in \mathbb{C} \setminus [a, +\infty). \tag{3.5}$$

If we now put

$$T(z) = e^{-n(l/2)\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{n(l/2)\sigma_3}, \tag{3.6}$$

where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the third Pauli matrix, then $T$ satisfies the Riemann-Hilbert problem

$$\begin{cases} T(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\ T_+(x) = T_-(x) J_T(x) \text{ for } x \in \mathbb{R}, \\ T(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty, \end{cases} \tag{3.7}$$

where

$$J_T(x) = \begin{cases} \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x < a, \\ \begin{pmatrix} e^{2n\phi_+(x)} & 1 \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} & \text{for } x \in (a,b), \\ \begin{pmatrix} 1 & e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x > b. \end{cases} \tag{3.8}$$

Since the inequality in (3.1) is strict for $x < a$ and $x > b$ we have that $\tilde{\phi}(x) > 0$ for $x < a$ and $\phi(x) > 0$ for $x > b$. Thus the jump matrices for $T$ on $(-\infty, a)$ and $(b, \infty)$ tend to the identity matrix as $n \to \infty$.

### 3.2 The Second Step: Transformation $T \mapsto S$

The second transformation is the so-called *opening of the lens* and it is based on the factorisation

$$\begin{pmatrix} e^{2n\phi_+(x)} & 1 \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_-(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_+(x)} & 1 \end{pmatrix} \tag{3.9}$$

of the jump matrix $J_T$ on $(a,b)$. The factorisation (3.9) allows us to split the jump on $(a,b)$ as shown in Figure 1.

We use $\Sigma_1$ and $\Sigma_2$ to denote the upper and lower lips of the lens, respectively. We define $S$ as follows:

- For $z$ outside the lens, we put $S = T$.
- For $z$ within the region enclosed by $\Sigma_1$ and $(a,b)$,

$$S = T \begin{pmatrix} 1 & 0 \\ -e^{2n\phi} & 1 \end{pmatrix}. \tag{3.10}$$

- For $z$ within the region enclosed by $\Sigma_1$ and $(a,b)$,

$$S = T \begin{pmatrix} 1 & 0 \\ e^{2n\phi} & 1 \end{pmatrix}. \tag{3.11}$$

Then $S$ satisfies the following Riemann-Hilbert problem:

$$\begin{cases} S(z) \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup \Sigma_1 \cup \Sigma_2) \\ S_+(z) = S_-(z) J_S(z) \text{ for } z \in \mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \\ S(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases} \tag{3.12}$$

where

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} \text{ for } z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } z \in (a,b), \\ \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(z)} \\ 0 & 1 \end{pmatrix} \text{ for } z < a, \\ \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} \text{ for } z > b, \end{cases} \tag{3.13}$$

We may (and do) assume that the lips of the lens are in the region where Re $\phi < 0$, so that the jump matrices for $S$ on $\Sigma_1$ and $\Sigma_2$ tend to the identity matrix as $n \to \infty$.

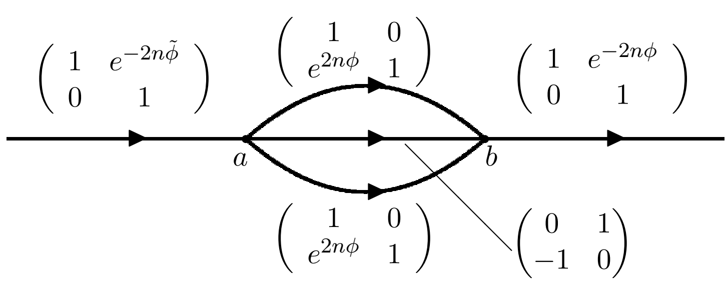

Figure 1: Jump matrices for $S$ after opening of the lens

### 3.3 The Third Step: Parametrix Away From Endpoints

The parametrix away from the branch points is a 'global solution' $N(z)$ satisfying the Riemann-Hilbert problem

$$\begin{cases} N(z) \text{ is analytic in } \mathbb{C} \setminus [a,b] \\ N_+(x) = N_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } x \in (a,b) \\ N(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases} \tag{3.14}$$

which has solution (see [5])

$$N(z) = \begin{pmatrix} \frac{\beta(z)+\beta^{-1}(z)}{2} & \frac{\beta(z)-\beta^{-1}(z)}{2i} \\ -\frac{\beta(z)-\beta^{-1}(z)}{2i} & \frac{\beta(z)+\beta^{-1}(z)}{2} \end{pmatrix} \tag{3.15}$$

where $\beta(z) = \left(\frac{z-b}{z-a}\right)^{\frac{1}{4}}$.

### 3.4 The Fourth Step: Parametrices Near Endpoints

Having constructed the 'global solution', the next step is finding 'local solutions' close to the endpoints $a$ and $b$. Near $b$, the local situation is described as in the left picture of Figure 2 with jump matrix

$$J_P(z) = J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} \text{ on } \Sigma_1 \cap U \text{ and } \Sigma_2 \cap U \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ on } (a,b) \cap U \\ \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} \text{ on } (b,\infty) \cap U \end{cases}$$

where $U$ is a (small) disk around $b$.

We therefore want to find a matrix function $P$, that solves

$$\begin{cases} P(z) \text{ is analytic on } U \setminus (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \\ P_+(z) = P_-(z) J_P(z) \text{ on } (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \cap U \\ P(z) = N(z)\left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ uniformly for } z \in \partial U \end{cases}$$

Then $P(z)e^{n\phi(z)\sigma_3}$ should have constant jumps on $(\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \cap U$, namely

$$\left(P(z)e^{n\phi(z)\sigma_3}\right)_+ = \left(P(z)e^{n\phi(z)\sigma_3}\right)_- \times \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ for } z \in (\Sigma_1 \cup \Sigma_2) \cap U \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } z \in (a,b) \cap U \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } z \in (b,\infty) \cap U \end{cases}$$

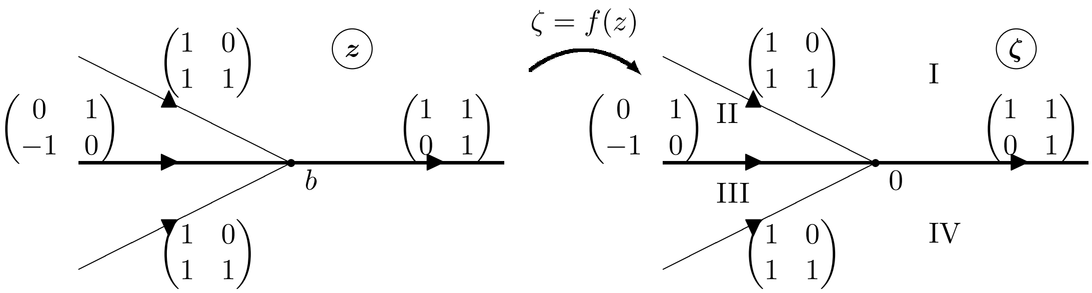


Figure 2: Mapping of neighbourhood of $b$ onto a neighbourhood of $f(b) = 0$

Shrinking $U$ if necessary, we have that

$$\zeta = f(z) = \left(\frac{3}{2}\phi(z)\right)^{2/3}$$

defines a conformal map from $U$ to a convex neighborhood of $\zeta = 0$. We may and do assume that the lips of the lens are taken so that $\Sigma_1 \cap U$ is mapped into $\arg\zeta = 2\pi/3$, and $\Sigma_2 \cap U$ is mapped into $\arg\zeta = 2\pi/3$, see Figure 2. Denoting the sectors in the $\zeta$-plane by I, II, III, IV as in Figure 2, and using the usual Airy function $\mathrm{Ai}(\zeta)$, we construct the Airy model solution $\Phi$ by

$$\Phi(\zeta) = \begin{cases} \begin{pmatrix} \mathrm{Ai}(\zeta) & \omega\mathrm{Ai}(\omega\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2\mathrm{Ai}'(\omega\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector IV} \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2\mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & -\omega\mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector I} \\ \begin{pmatrix} -\omega\mathrm{Ai}(\omega\zeta) & -\omega^2\mathrm{Ai}(\omega^2\zeta) \\ -\omega^2\mathrm{Ai}'(\omega\zeta) & -\omega\mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector II} \\ \begin{pmatrix} -\omega^2\mathrm{Ai}(\omega^2\zeta) & \omega\mathrm{Ai}(\omega\zeta) \\ -\omega\mathrm{Ai}'(\omega^2\zeta) & \omega^2\mathrm{Ai}'(\omega\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector III} \end{cases}$$

which has the jump matrices in the $\zeta$-plane indicated in the right side of Figure 2.

Then for any analytic prefactor $E_n(z)$ we have that

$$P(z) = E_n(z)\Phi(n^{\frac{2}{3}}f(z))e^{n\phi(z)\sigma_3} \tag{3.16}$$

has the required jump matrices $J_P$. If we choose

$$E_n = \sqrt{\pi}N(z)\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\left(n^{2/3}f(z)\right)^{\sigma_3/4} \tag{3.17}$$

then the matching condition $P(z) = N(z)(I+\mathcal{O}(1/n))$ as $n \to \infty$ for $z \in \partial U$, is satisfied as well, see e.g. [3, 5, 7] for further detail.

A similar construction yields a parametrix $\tilde{P}$ in a small disc $\tilde{U}$ around $a$. One can see that $\tilde{P}$ can be obtained by taking $P$ and interchanging $a$ and $b$ and conjugating with $\sigma_3$.

### 3.5 The Fifth Step: Transformation $S \mapsto R$

Using the parametrices $N$, $P$, and $\tilde{P}$, we define the third transformation $S \mapsto R$ as follows

$$R(z) = \begin{cases} S(z)N(z)^{-1} & \text{for } z \in \mathbb{C} \setminus \overline{(U \cup \tilde{U})} \\ S(z)P(z)^{-1} & \text{for } z \in U \\ S(z)\tilde{P}(z)^{-1} & \text{for } z \in \tilde{U} \end{cases} \tag{3.18}$$

Then $R$ has no jump on $[a,b] \setminus \overline{(U \cup \tilde{U})}$, as the jumps of $S$ and $N^{-1}$ cancel out. In $U$ and $\tilde{U}$ the jumps of $S$ cancel out with the jumps of $P$ and $\tilde{P}$, leaving only jumps for $R$ on the contour $\Sigma_R$ shown in Figure 3.

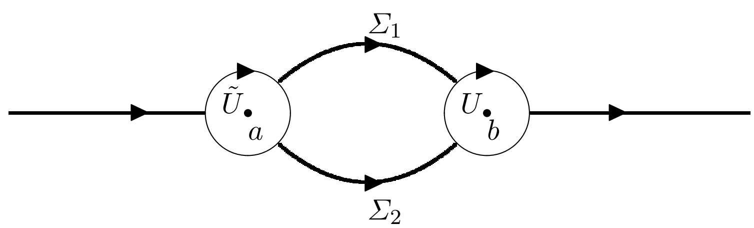


Figure 3: Contour $\Sigma_R$ for the Riemann-Hilbert problem for $R$

The Riemann-Hilbert problem for $R$ is

$$\begin{cases} R(z) \text{ is analytic on } \mathbb{C} \setminus \Sigma_R \\ R_+(z) = R_-(z)J_R(z) \text{ for } z \in \Sigma_R \\ R(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \end{cases}$$

where

$$J_R(z) = \begin{cases} N(z)J_S(z)N(z)^{-1} \text{ for } z \in \Sigma_R \setminus (\partial U \cup \partial\tilde{U}) \\ P(z)N(z)^{-1} \text{ for } z \in \partial U \\ \tilde{P}(z)N(z)^{-1} \text{ for } z \in \partial\tilde{U} \end{cases}$$

The jump matrices $J_R(z) = N(z)J_S(z)N(z)^{-1}$ tend to the identity matrix at an exponential rate as $n \to \infty$. The jump matrices on $\partial U$ and $\partial\tilde{U}$ tend to the identity matrix but at a slower rate of $1/n$ as $n \to \infty$. The

precise form is obtained from the asymptotic expansion of the Airy function as $z \to \infty$, $-\pi < \arg z < \pi$, (see [13])

$$\mathrm{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}z^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(3k+\frac{1}{2}\right)}{9^k (2k)! \Gamma\left(\frac{1}{2}\right)} \frac{1}{z^{\frac{3}{2}k}} \tag{3.19}$$

and the corresponding asymptotic expansion for $\mathrm{Ai}'(z)$. Using these facts in the parametrix $P$ we find an asymptotic expansion for the jump of $R$ on $\partial U$

$$J_R(z) = P(z)N(z)^{-1} \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} \Delta_k(z) \tag{3.20}$$

where

$$\begin{aligned} \Delta_k(z) =& \frac{1}{\sqrt{\pi}} \left( \frac{\Gamma\left(3k+\frac{1}{2}\right)}{9^k(2k)!} - \frac{\Gamma\left(3k-\frac{3}{2}\right)}{4\cdot 9^{k-1}(2(k-1))!} \right) \frac{1}{\left(\frac{3}{2}\phi(z)\right)^k} I \\ & - \frac{1}{4\sqrt{\pi}} \frac{\Gamma\left(3k-\frac{3}{2}\right)}{9^{k-1}(2(k-1))!} \frac{1}{\left(\frac{3}{2}\phi(z)\right)^k} \sigma_2 \qquad \text{for } k \text{ even} \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \Delta_k(z) =& -\frac{\beta(z)^2}{\left(\frac{3}{2}\phi(z)\right)^k} \frac{1}{2\sqrt{\pi}} \left( \frac{\Gamma\left(3k+\frac{1}{2}\right)}{9^k(2k)!} - \frac{\Gamma\left(3k-\frac{3}{2}\right)}{2\cdot 9^{k-1}(2(k-1))!} \right) (\sigma_3 + i\sigma_1) \\ & - \frac{\beta(z)^{-2}}{\left(\frac{3}{2}\phi(z)\right)^k} \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(3k+\frac{1}{2}\right)}{9^k(2k)!} (\sigma_3 - i\sigma_1) \qquad \text{for } k \text{ odd} \end{aligned} \tag{3.22}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.23}$$

are the Pauli matrices.

A similar expansion

$$J_R(z) = \tilde{P}(z)N(z)^{-1} \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} \tilde{\Delta}_k(z) \tag{3.24}$$

holds for the jump matrix on $\partial\tilde{U}$.

As a result we find by the methods of [7], see also [14, Lemma 8.3],

**Lemma 3.1.** There exist matrix valued functions $R_k(z)$ with the property that for every $l \in \mathbb{N}$, there exist constants $C > 0$ and $r > 0$ such that for every $z$ with $|z| \geq r$,

$$\left\| R(z) - I - \sum_{k=1}^{l} \frac{R_k(z)}{n^k} \right\| \leq \frac{C}{|z| n^{l+1}} \tag{3.25}$$

So we write

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} R_k(z) \tag{3.26}$$

From (3.26), (3.20) and (3.24) and the Riemann-Hilbert problem for $R$, we find an additive Riemann-Hilbert problem for $R_k(z)$,

$$\begin{cases} R_k(z) \text{ is analytic on } \mathbb{C} \setminus (\partial U \cup \partial \tilde{U}) \\ R_{k+}(z) = R_{k-}(z) + \sum\limits_{l=0}^{k-1} R_{l-}(z) \Delta_{k-l}(z) \text{ for } z \in \partial U \\ R_{k+}(z) = R_{k-}(z) + \sum\limits_{l=0}^{k-1} R_{l-}(z) \tilde{\Delta}_{k-l}(z) \text{ for } z \in \partial \tilde{U} \\ R_k(z) = \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty \end{cases} \tag{3.27}$$

where $R_0(z) = I$. These Riemann-Hilbert problems can be successively solved using the Sokhotskii-Plemelj formula, or using a technique based on Laurent series expansion as in [14].

# 4 Proof of Theorem 1.1

For the proof of (1.4) we do not need to compute the explicit forms of the $R_k$'s. However, we need to know that they have the following structure. Recall that the Pauli matrices are given in (3.23).

**Lemma 4.1.** For $k$ odd, $R_k(z)$ is a linear combination of $\sigma_1$ and $\sigma_3$ and for $k$ even, $R_k(z)$ is a linear combination of $I$ and $\sigma_2$.

*Proof.* For $k = 1$, we know because of (3.27) that $R_{1+} = R_{1-} + \Delta_1$ on $\partial U$ and $R_{1+} = R_{1-} + \tilde{\Delta}_1$ on $\tilde{\partial} U$. As $\Delta_1$, $\tilde{\Delta}_1 \in \operatorname{span}\{\sigma_1, \sigma_3\}$ on account of (3.22), $R_1(z)$ must be a linear combination of $\sigma_1$ and $\sigma_3$ as well.

Let $k \geq 1$ and once more observe (3.27). If $k$ is odd, then again by (3.22) $\Delta_k$, $\tilde{\Delta}_k \in \operatorname{span}\{\sigma_1, \sigma_3\}$ and using induction on $k$, for every $l < k$, $R_{l-}(z)\Delta_{k-l}(z)$ and $R_{l-}(z)\tilde{\Delta}_{k-l}(z)$ are products of a linear combination of

$\sigma_1$ and $\sigma_3$ and a linear combination of $I$ and $\sigma_2$ (see also (3.21)–(3.22)), which results in a linear combination of $\sigma_1$ and $\sigma_3$. Thus all terms in the (additive) jump for $R_k$ on $\partial U$ and on $\partial \tilde{U}$ are in the span of $\sigma_1$ and $\sigma_3$, and it follows that $R_k \in \operatorname{span}\{\sigma_1, \sigma_3\}$ if $k$ is odd.

If $k$ is even, then by induction, where we use again ((3.21)–(3.22), we have that $R_{l-}(z)\Delta_{k-l}(z)$ and $R_{l-}(z)\tilde{\Delta}_{k-l}(z)$ are either products of two linear combinations of $I$ and $\sigma_2$ (in case $l$ is even), or products of two linear combinations of $\sigma_1$ and $\sigma_3$ (in case $l$ is odd). In both cases we find that $R_{l-}(z)\Delta_{k-l}(z)$ and $R_{l-}(z)\tilde{\Delta}_{k-l}(z)$ are linear combinations of $I$ and $\sigma_2$, which implies that $R_k \in \operatorname{span}\{I, \sigma_2\}$ if $k$ is even. □

Now we can finally prove our main result.

*Proof of Theorem 1.1.* We start from the expressions (2.4) and (2.5) for $a_{n,n}$ and $b_{n,n}$ in terms of the solution of the Riemann-Hilbert problem for $Y$. Following the transformations $Y \mapsto T \mapsto S$, we find that

$$a_{n,n} = (S_1)_{12}\,(S_1)_{21} \tag{4.1}$$

and

$$b_{n,n} = \frac{(S_2)_{12}}{(S_1)_{12}} - (S_1)_{22} \tag{4.2}$$

where $S_1$ and $S_2$ are the terms in the expansion of $S(z)$ as $z \to \infty$,

$$S(z) = I + \frac{1}{z}S_1 + \frac{1}{z^2}S_2 + \mathcal{O}\left(\frac{1}{z^3}\right).$$

To obtain (4.2) we use that $g(z) = \log z + \mathcal{O}(1/z)$, see also [10].

By (3.18), we know that $S(z) = R(z)N(z)$ for $|z|$ large enough, so we need the first terms in the expansions of $N(z)$ and $R(z)$ as $z \to \infty$. From (3.15) we have

$$\begin{aligned} N(z) &= \frac{\beta(z) + \beta(z)^{-1}}{2} I + \frac{\beta(z) - \beta(z)^{-1}}{2}\sigma_2 \\ &= I - \frac{(b-a)}{4}\sigma_2 \frac{1}{z} + \left(\frac{(b-a)^2}{32} I - \frac{b^2-a^2}{8}\sigma_2\right)\frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \end{aligned} \tag{4.3}$$

and from Lemma 4.1

$$
\begin{aligned}
R(z) = I &+ \frac{1}{z}\left(\sum_{m \text{ odd}} \frac{1}{n^m}\left(R_{m1\sigma_1}\sigma_1 + R_{m1\sigma_3}\sigma_3\right) + \sum_{m \text{ even}} \frac{1}{n^m}\left(R_{m1I}I + R_{m1\sigma_2}\sigma_2\right)\right) \\
&+ \frac{1}{z^2}\left(\sum_{m \text{ odd}} \frac{1}{n^m}\left(R_{m2\sigma_1}\sigma_1 + R_{m2\sigma_3}\sigma_3\right) + \sum_{m \text{ even}} \frac{1}{n^m}\left(R_{m2I}I + R_{m2\sigma_2}\sigma_2\right)\right) \\
&+ \mathcal{O}\left(\frac{1}{z^3}\right)
\end{aligned}
\tag{4.4}
$$

where the constants $R_{mjI}$, $R_{mj\sigma_k}$, for $m \in \mathbb{N}$, $j = 1, 2$, and $k = 1, 2, 3$ are such that $R_{mjI}I + \sum_{k=1}^{3} R_{mj\sigma_k}\sigma_k$ is the coefficient of $z^{-j}$ in the Laurent expansion of $R_m(z)$ around $z = \infty$.

Therefore, by (4.3) and (4.4),

$$
\begin{aligned}
S(z) = &R(z)N(z) \sim I + \frac{1}{z}\left(-\frac{(b-a)}{4}\sigma_2 + \sum_{m \text{ odd}} \frac{1}{n^m}\left(R_{m1\sigma_1}\sigma_1 + R_{m1\sigma_3}\sigma_3\right)\right. \\
&\left. + \sum_{m \text{ even}} \frac{1}{n^m}\left(R_{m1I}I + R_{m1\sigma_2}\sigma_2\right)\right) \\
&+ \frac{1}{z^2}\left(\frac{(b-a)^2}{32}I - \frac{b^2-a^2}{8}\sigma_2 + \sum_{m \text{ odd}} \frac{1}{n^m}\left(\left(R_{m2\sigma_1} + i\frac{b-a}{4}R_{m1\sigma_3}\right)\sigma_1\right.\right. \\
&\left. + \left(R_{m2\sigma_3} - i\frac{b-a}{4}R_{m1\sigma_1}\right)\sigma_3\right) + \sum_{m \text{ even}} \frac{1}{n^m}\left(\left(R_{m2I} - \frac{b-a}{4}R_{m1\sigma_2}\right)I\right. \\
&\left.\left. + \left(R_{m2\sigma_2} - \frac{b-a}{4}R_{m1I}\right)\sigma_2\right)\right) + \mathcal{O}\left(\frac{1}{z^3}\right)
\end{aligned}
\tag{4.5}
$$

which implies that

$$
(S_1)_{12} \sim \frac{b-a}{4}i + \sum_{m \text{ odd}} \frac{1}{n^m}R_{m1\sigma_1} - i\sum_{m \text{ even}} \frac{1}{n^m}R_{m1\sigma_2} \tag{4.6}
$$

and

$$
(S_1)_{21} \sim -\frac{b-a}{4}i + \sum_{m \text{ odd}} \frac{1}{n^m}R_{m1\sigma_1} + i\sum_{m \text{ even}} \frac{1}{n^m}R_{m1\sigma_2} \tag{4.7}
$$

Inserting (4.6) and (4.7) into (4.1) then finally gives

$$
a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{\alpha_{2m}}{n^{2m}}
$$

for certain constants $\alpha_{2m}$.

Similar to (4.6) and (4.7) we have that $(S_2)_{12}$ and $(S_1)_{22}$ have asymptotic expansions in powers of $1/n$. From the expansion (4.5) for $S$, we see

$$\begin{aligned}(S_2)_{12} \sim & \frac{b^2-a^2}{8} i + \sum_{m \text{ odd}} \frac{1}{n^m}\left(\frac{b-a}{4} i R_{m1\sigma_3} + R_{m2\sigma_1}\right) \\ & + \sum_{m \text{ even}} \frac{1}{n^m} i \left(\frac{b-a}{4} R_{m1I} - R_{m2\sigma_2}\right)\end{aligned}$$

and

$$(S_1)_{22} \sim - \sum_{m \text{ odd}} \frac{1}{n^m} R_{m1\sigma_3} + \sum_{m \text{ even}} \frac{1}{n^m} R_{m1I}$$

From (4.2) it then follows that

$$b_{n,n} \sim \sum_{m=0}^{\infty} \frac{\beta_m}{n^m} \tag{4.8}$$

where $\beta_0 = \frac{b+a}{2}$ and

$$\beta_1 = 2R_{11\sigma_3} - \frac{4}{b-a} i R_{12\sigma_1} + \frac{2(b+a)}{b-a} i R_{11\sigma_1}. \tag{4.9}$$

Our final task is to further evaluate the right-hand side of (4.9). As in [14], we have that $\Delta_1$ is meromorphic in a neighborhood of $b$ with a pole in $b$. Indeed, if we write

$$\frac{\beta(z)^{-2}}{\phi(z)} = (z-b)^{-2} \sum_{m=0}^{\infty} B_m (z-b)^m, \qquad B_0 = \frac{3}{2\pi h(b)}, \tag{4.10}$$

and use (3.22), then we find for $z$ in a neighborhood of $b$,

$$\begin{aligned}\Delta_1(z) = & \left(-\frac{5B_1}{144}(\sigma_3 - i\sigma_1) + \frac{7B_0}{144(b-a)}(\sigma_3 + i\sigma_1)\right) \frac{1}{z-b} \\ & - \frac{5B_0}{144}(\sigma_3 - i\sigma_1) \frac{1}{(z-b)^2} + \mathcal{O}(1).\end{aligned} \tag{4.11}$$

Similarly, for $z$ in a neighborhood of $a$, we have

$$\frac{\beta(z)^{2}}{\tilde{\phi}(z)} = (z-a)^{-2} \sum_{m=0}^{\infty} A_m (z-a)^m, \qquad A_0 = \frac{3}{2\pi h(a)}, \tag{4.12}$$

and

$$\tilde{\Delta}_1(z) = \left( -\frac{5A_1}{144}\left(\sigma_3 + i\sigma_1\right) - \frac{7A_0}{144(b-a)}\left(\sigma_3 - i\sigma_1\right)\right)\frac{1}{z-a} \\ - \frac{5A_0}{144}\left(\sigma_3 + i\sigma_1\right)\frac{1}{(z-a)^2} + \mathcal{O}\left(1\right). \tag{4.13}$$

As in [14] we have that $R_1(z)$ for $z \in \mathbb{C} \setminus \overline{U \cup \tilde{U}}$ is equal to the sum of the Laurent parts of (4.11) and (4.13). Expanding $R_1(z)$ as $z \to \infty$, we then get

$$R_1(z) = R_{11}\frac{1}{z} + R_{12}\frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \quad \text{as } z \to \infty,$$

where

$$\begin{aligned} R_{11} = &- \frac{5A_1}{144}\left(\sigma_3 + i\sigma_1\right) - \frac{7A_0}{144(b-a)}\left(\sigma_3 - i\sigma_1\right) \\ &- \frac{5B_1}{144}\left(\sigma_3 - i\sigma_1\right) + \frac{7B_0}{144(b-a)}\left(\sigma_3 + i\sigma_1\right) \\ R_{12} = &- \frac{5aA_1}{144}\left(\sigma_3 + i\sigma_1\right) - \frac{7aA_0}{144(b-a)}\left(\sigma_3 - i\sigma_1\right) \\ &- \frac{5bB_1}{144}\left(\sigma_3 - i\sigma_1\right) + \frac{7bB_0}{144(b-a)}\left(\sigma_3 + i\sigma_1\right) \\ &- \frac{5A_0}{144}\left(\sigma_3 + i\sigma_1\right) - \frac{5B_0}{144}\left(\sigma_3 - i\sigma_1\right). \end{aligned}$$

Thus

$$R_{11\sigma_3} = -\frac{5(A_1+B_1)}{144} - \frac{7(A_0-B_0)}{144(b-a)}, \tag{4.14}$$

$$R_{11\sigma_1} = -i\frac{5(A_1-B_1)}{144} + i\frac{7(A_0+B_0)}{144(b-a)}, \tag{4.15}$$

$$R_{12\sigma_1} = -i\frac{5(aA_1-bB_1)}{144} + i\frac{7(aA_0+bB_0)}{144(b-a)} - i\frac{5(A_0-B_0)}{144}. \tag{4.16}$$

Inserting (4.14)–(4.16) into (4.9), we find after straightforward calculations that $A_1$ and $B_1$ fully disappear and that (4.9) reduces to

$$\beta_1 = \frac{B_0 - A_0}{3(b-a)}.$$

Using the explicit formulas for $A_0$ and $B_0$ given in (4.10) and (4.12), we arrive at (1.5), which completes the proof of Theorem 1.1. $\square$

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